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REGULATION CIRCUITS WITH LINKS EACH POSSESSING SEVERAL DEGREES OF FREEDOM

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1. Matrix Equations of Circuit Links.

Regulated systems are usually simple or branched closed circuits in which each link affect the next link, while the reverse action [feedback] may be neglected. The links possess one or several degrees of freedom; in the latter case any one preferred coordinate may be chosen for each link, and by eliminating the remaining coordinates from the system of differential equations of the link, (which equations are assumed to be linear and with constant coefficients) we obtain a single high-order equation. Because, however, this is frequently associated with complicated transformations and the equations lose direct physical significance, it seems advisable to operate with the matrix equations corresponding to the system of scalar equations of the link.

For instance, let the equation of α_1 have the form:

$$\sum_{k=1}^{m_1} f_{jk}^{(1)}(D)\, y_k^{(1)} = \sum_{k'=1}^{m} e_{jk'}^{(1)}(D)\, x_{k'} \qquad (j = 1, \dots, m_1)$$

Here $y_1^{(1)}, \dots, y_{m_1}^{(1)}$ are the coordinates of the link; $x_1, \dots, x_{m_x}$ are functions representing the reactions external to the link; D is the differential operator d/dt and finally $f_{jk}^{(1)}(D)$ and $e_{jk}^{(1)}(D)$ are polynomials in D. Let us introduce the matrix columns

$$y_1 = \left\| \begin{matrix} y_1^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ y_{m_1}^{(1)} \end{matrix} \right\| \quad (m_1 \times 1), \qquad x = \left\| \begin{matrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_{m_x} \end{matrix} \right\| \quad (m_x \times 1)$$

and polynomial matrices

$$f_1(D) = \left\| f_{jk}^{(1)}(D) \right\| \qquad (m_1 \times m_1) \qquad e_1(D) = \left\| e_{jk}^{(1)}(D) \right\| (m_1 \times m_x)$$

(All operations, terminology and designations of matrix calculations used here and below are either universally adopted or explained in (1). The system of equations of the link may be replaced by the matrix equation

$$f_1(D)y_1 = e_1(D)x$$

Let us have a second link α_2 with a matrix coordinate y_2 of the type $m_2 \times 1$, and let two links be connected in series; that is, the first is affected only by an action x external to the system of two links, and the second link is affected only by the action of the first link (Figure 1). The matrix equation of the second link will have the form:

$$f_2(D)y_2 = e_2(D)y_1$$

where $f_2(D)$, $e_2(D)$ are polynomial matrices of the types $m_2 \times m_2$ and $m_2 \times m_1$. Assuming the matrices $f_1(D)$ and $f_2(D)$ to have no singularities and solving the equations formally for y_1 and y_2 by dividing the left term by $f_1(D)$ and $f_2(D)$ we find

$$y_1 = X_1(D)x \qquad y_2 = X_2(D)y_1$$

where each matrix

$$X_1(D) = f_1(D) \diagdown e_1(D) \qquad X_2(D) = f_2(D) \diagdown e_2(D)$$

designates the transmittance of the corresponding link. The line inclined to the left indicates that the divisor is from left of the line.

Eliminating y_1 we have

$$y_2 = X(D)x$$

The matrix

$$X(D) = X_2(D)X_1(D)$$

plays the role of transmittance in the system of two links.

Hence, the transmittances of links connected in series are multiplied. ere the order of the factors in the product counting from right to left corresponds to the order of succession of the links in the circuit relative to unidirectional action.

If each link has one degree of freedom and $m_x = 1$, then the transmittances $X_1(D)$ and $X_2(D)$ act as scalars and the order of their multiplication is immaterial.

The rule established is a generalization of the corresponding rule for the calculation of four-terminal networks (2) as in the theory of wire communications in radio engineering.

If D is replaced by iw, as is done in a-c theory, then the transmittances are converted into the corresponding matrix frequency characteristics.

We note that the schematic representation of the structure of a regulated system as a circuit contains an element arbitrary to the extent that in its analysis one can combine adjoining links into one block, by considering it as a single link. Therefore if any regulated system cannot be represented as single-loop circuit with more than two links, it may always be reduced to a two-link closed loop. To do this one must merely separate in some way or other all coordinates and scalar equations into two groups and relate one group to the first link and the second group to the second link.

2. Single-Loop Closed Circuit

Assume that we have a single-loop closed regulation circuit consisting of n links $\alpha_1, \ldots, \alpha_n$ (Figure 2). The first link may be the regulation field. (Note: This term may be used instead "object of regulation", because one and the same object may be subjected to regulation with respect to its various physical properties; for example, pressure and temperature). The remaining links are the various elements of the regulator. Let the number of degrees of freedom of link $_j$ be m_j. An impulse is applied to the first link and a command is impressed on link α_{r+1}.

The matrix equations of a system of n links have the form:

$$f_1(D)y_1 = e_1(D)y_n + e_{1z}(D)_z$$

$$f_2(D)y_2 = e_2(D)y_1$$

$$\cdots\cdots\cdots\cdots$$

$$f_r(D)y_r = e_r(D)y_{r-1}$$

$$f_{r+1}(D)y_{r+1} = -e_{r+1}(D)y_r + e_{r+1,x}(D)x$$

$$\cdots\cdots\cdots\cdots$$

$$f_n(D)y_n = e_n(D)y_{n-1}$$

Here the matrices y_j, x, z, of type $m_j \times 1$, $m_x \times 1$, $m_z \times 1$, characterize respectively: state command, load variation and disturbance in the links. The functions $f(D)$ are nonsingular quadratic polynomial matrices of order m_j, and $e_1(D), \ldots, e_n(D)$, $e_{r+1,x}(D)$, $e_{1z}(D)$ in general are square polynomial matrices of types $m_1 \times m_n, \ldots, m_n \times m_{n-1}$, $m_{r+1} \times m_x$, $m_1 \times m_z$.

The designations are chosen in such a way that in one equation - namely, before the operational matrix $e_{r+1}(D)$ -- the sign should be minus; this expresses the fact that the regulator must counteract any disturbance of the regime desired.

Starting with the second let us multiply the matrix equations on the left by the matrices following

$$-e_1(D)f_n^{-1}(D)e_n(D)\ldots f_2^{-1}(D)$$

$$\cdots\cdots\cdots\cdots$$

$$-e_1(D)f_n^{-1}(D)e_n(D)\ldots f_r^{-1}(D)$$

$$e_1(D)f_n^{-1}(D)e_n(D)\ldots f_{r+1}^{-1}(D)$$

$$\cdots\cdots\cdots\cdots$$

$$e_1(D)f_n^{-1}(D)$$

and add to the first. After reduction we get

$$[f_1(D) + e_1(D)f_n^{-1}(D)e_n(D) \dots f_2^{-1}(D)e_2(D)]y_1 =$$

$$= e_1(D)f_n^{-1}(D)e_n(D)\, f_{r+1}^{-1}(D)e_{r+1,x}(D)x + e_{1_z}(D)z \qquad (2.2)$$

Setting

$$\Delta_j(D) = \det f_j(D), \qquad F_j(D) = \operatorname{adj} f_j(D)$$

where the symbol "adj" designates the adjoint matrix, let us introduce the transmittances of unidirectional actions between adjacent links

$$X_j(D) = f_j(D) \diagdown e_j(D) = \frac{F_j(D)e_j(D)}{\Delta_j(D)} \qquad (j = 1, \dots, n) \qquad (2.3)$$

and the transmittances of command and impulse

$$X_{r+1,x}(D) = f_{r+1}(D) \diagdown e_{r+1,x}(D) = \frac{F_{r+1}(D)e_{r+1,x}(D)}{\Delta_{r+1}(D)}$$

$$X_{1_z}(D) = f_1(D) \diagdown e_{1_z}(D) = \frac{F_1(D)e_{1_z}(D)}{\Delta_1(D)} \qquad (2.4)$$

Then after division from the left by $f_1(D)$, the equation (2.2) will take the form:

$$[E_{m1} + X(D)]y_1 = e_x(D)x + e_z(D)Z$$

where

$$X(D) = X_1(D)X_n(D) \dots X_2(D)$$

$$e_x(D) = X_1(D)X_n(D) \dots X_{r+1,x}(D), \quad e_z(D) = X_{1_z}(D) \qquad (2.5)$$

or

$$y_1 = X_x(D)x + X_z(D)Z \qquad (2.6)$$

while

$$X_x(D) = [E_{m1} + X(D)] \diagdown e_x(D), \quad X_z(D) = [E_{m1} + X(D)] \diagdown e_z(D) \qquad (2.7)$$

The square matrix

$$X(D) = \left\| X_{jk}(D) \right\| \qquad (m_1 \times m_1) \tag{2.8}$$

represents the transmittance of the open circuit $_2 \cdot \cdot \cdot \; _n \; _1$ which circuit is obtained by rupture of the connection between the first and second link, as shown in the schematic drawing. If $x = z = 0$, then by supplying, for example, to the input of the first link α_2 of the circuit instead of y_1, some sinusoidal signal $u(t)$, we shall et at the output of the last link α_1 the signal $- X(D)u(t)$ (the minus sign was obtained because of its presence in the $(r + 1)$th equation). If $[y_j^{(1)}]$ designates the dimensions of the scalar coordinate $y_j^{(1)}$, then the dimension $X_{jk}(D)$ obviously will be $[y_j^{(1)}]/[y_k^{(1)}]$; if in particular all coordinates $y_j^{(1)}$ are dimensionless, then all $X_{jk}(D)$ also will be dimensionless.

Generalizing somewhat Bode's terminology (3), we shall designate the matrix $E_{m1} + X(D)$, the "recurrent difference," since $[E_{m_1} + X(D)]u(t)$ is the difference between the signals $u(t)$ and $-X(D)u(t)$.

The matrices $X_x(D)$, $X_z(D)$ represent the transmittances of the regulation field with the regulator connected relative to command and disturbances.

That part of the circuit between regulation field and position of command input which part consists of the links $\alpha_2, \ldots, \alpha_r$ may be called the feedback since one records with its aid the state of the regulation field and the transmission of the corresponding signal for comparison with the command, which comparison takes place in the link α_{r+1}. If the feedback is severed, then obviously

$$y_1 = e_x(D)x + e_z(D)Z \tag{2.9}$$

that is, the transmittances $X_x(D)$, $X_z(D)$ will be reduced to $e_x(D)$, $e_z(D)$. The term $X(D)$ in the denominators of the right-hand terms of formulas (2.7) will therefore express the influence of feedback. Equation (2.9) characterizes the control of an open-circuit type, and equation (2.6) characterizes automatic regulation.

If $X_x(0)$ is finite, then we have for x = const, $z = 0$ the static solution

$$y_{1_x} = X_x(0)x;$$

If in this case the conditions of asymptotic stability are satisfied, then this solution characterizes the state to which the regulation field tends in response to a constant command; the discrepancy $y_{1_x} - y_1$ tends here to zero.

In a similar way, if $X_z(0)$ is finite, then for $x = 0$, z = const we obtain a partial solution

$$y_{1_z} = X_z(0)z$$

characterizing the static reaction of the regulation field to a constant disturbance; for instance, to a decrease in load of a steam engine by a constant amount. If

$$X_z(0) = 0,$$

then the static reaction equals zero. Sudden imposition of a constant disturbance produces a transient process, after the termination of which the regulation field returns to its previous state; hence we have isodromic regulation.

Also those systems in which not all, but only some of the most important coordinates have zero static values for constant disturbance may be called isodromic regulation systems; but this case may be reduced to the previous one, if we narrow the definition of a regulation field and assume that it is determined only by the values of those coordinates of interest to us.

By making the substituion $D \to iw$ we may by the usual methods convert the transmittances entering in the previous equations to the corresponding frequency characteristics. These characteristics may be found not only by computation, but also by experiments for which it is necessary alternately to feed sinusoidal impulses to each scalar element of matrix coordinates at input and to measure the corresponding amplitudes and phases of the matrix elements coordinates at output. The characteristics $X_x(iw)$, $X_z(iw)$ determine the regimes of the stationary sinusoidal oscillations for sinusoidal command x and sinusoidal disturbance z.

If we replace in (2.6) the "undetermined" operator D by the Heavyside-Carston operator p, we shall obtain an equation of the transient process in the expression for the cases with zero initial values. In the cases of non-zero initial values it is necessary first to eliminate the denominators and to add, according to well known rules of operational calculus, the terms corresponding to the initial values. The non-equivalence of the initial and the transformed equations in the sense of, for instance, Luzin's definitions (4) is immaterial, if we correctly take into consideration the connection among the initial values of unknowns and their derivatives that enter the original and transformed equations. The application of the obtained equations to the calculation of the characteristic determinant and investigation of the stability of natural oscillations is discussed in sections 3 and 6 below.

In particular some or all links α_j may have one degree of freedom, so that $m_j = 1$; then the corresponding $f_j(D)$ will be scalar polynomials. The polynomial matrix $e_j(D)$ and the transmittance $X_j(D)$ will be scalars only in the case where both adjacent links α_{j-1}, α_j have one degree of freedom (at $j = 1$ this should hold relative to $\quad_n$ and $\quad_1$); similar remarks may be made relative to the matrices

$$e_{r+1}, x(D), e_{1_z}(D), X_{r+1}, x(D), X_{1_z}(D)$$

designating

$$\ell(D) = X(D) \setminus e_x(D) \tag{2.10}$$

we can write

$$e_x(D) = X(D)\, \ell(D)$$

$$X_x(D) = [E_{m_1} + X(D)] \setminus X(D)\, \ell(D) \tag{2.11}$$

In the elements of the matrix $E_{m_1} + X(D)$ let us mark the terms containing the degree D that is the highest for each column, and let us assume that the determinant of coefficients of these terms is different from zero. Next let us consider regulated systems for which within a certain band of frequencies the matrices X(i) are large. In this case the matrix

$$X_z(i\omega) = [E_{m_1} + X(i\omega)]^{-1} e_z(i\omega) = \frac{\operatorname{adj}[E_{m_1} + X(i\omega)]}{\det [E_{m_1} + X(i\omega)]} e_z(i\omega)$$

is small, and the distrubances hardly felt. As for the matrix

$$X_x(i\omega) = [E_{m_1} + X(i\omega)]^{-1} X(i\omega)\,\ell(i\omega) =$$

$$= [E_{m_1} + X(i\omega)]^{-1} [E_{m_1} + X(i\omega) - E_{m_1}]\,\ell(i\omega) =$$

$$= E_{m_1} - [E_{m_1} + X(i\omega)]^{-1} E_{m_1}\,\ell(i\omega),$$

it approaches $\ell(i\omega)$ and therefore in the considered band the approximate value of the frequency characteristic of command is $\ell(i\omega)$. In order to present this case more concretely, let us assume that $m_1 = m_r = m_x$ and e_{r+1}, $x(D) = e_{r+1}(D)$ in consequence of which the transmittance of the feedback is

$$k(D) = X_r(D) \ldots X_2(D) \tag{2.12}$$

becomes a square matrix of the m-th order, as well as the matrix

$$e_x(D) = X_1(D)X_n(D) \ldots X_{r+1}(D) \tag{2.13}$$

thus

$$X(D) = e_x(D)k(D) = X(D)\ (D)k(D), \qquad (D) = k^{-1}(D)$$

so that the frequency characteristic of command is in the main determined by the feedback; in other words, the response of the system to the command depends only on the tuning of the measuring, amplifying, transmitting, digital and controlling elements that constitute the feedback line.

3. The Characteristic Equation of the Single-Loop Closed Circuit

Equation (2.2) was obtained from the given equations (2.1) by term-by term addition of the first to the others which others are multiplied on at the left by certain matrices. The results is elementary transformations of the third type in the sense of (1), section 1.7, performed on the rows of a lumped form of the operational matrix; that is, the matrix of operators in the unknowns.

Therefore if we replace the first of the given equations by equation (2.2) and leave the others without change, the operational matrix of the obtained system, which may be represented in the form

$$\left\| \begin{array}{cccccc} f_1(D) + e_1(D)X_n(D) \ldots X_2(D) & 0 & \ldots 0 & 0 \\ -e_2(D) & f_2(D) & \ldots 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & -e_n(D) & f_n(D) \end{array} \right\|$$

has the same determinant $\Lambda(D)$ as the operational matrix $f(D)$ of the original. Making use of Laplace's theorem we obtain

$$\Lambda(D) = \det [f_1(D) + e_1(D)X_n(D) \ldots X_2(D)] \det f_2(D) \ldots \det f_n(D) =$$
$$= \det f_1(D)[E_{m_1} + X(D)]\ \Lambda_2(D) \ldots \Lambda_n(D)$$

or

$$\Lambda(D) = \Lambda_0(D) \det [E_{m_1} + X(D)] \tag{3.1}$$

where the magnitude

$$\Lambda_0(D) = \Lambda_1(D) \ldots \Lambda_n(D) \tag{3.2}$$

represents the characteristic determinant of the open circuit.

Developing the determinant of the recurrent difference

$$E_{m_1} + X(D) = \left\| \begin{array}{ccc} 1 + X_{11}(D) & \ldots & X_{1m_1}(D) \\ \ldots & \ldots & \ldots \\ X_{m_1}(D) & \ldots & 1 + X_{m_1m_1}(D) \end{array} \right\|$$

we represent it in the form

$$\det [E_{m_1} + X(D)] = 1 + K(D) \tag{3.3}$$

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where

$$K(D) = \sum_i X_{ii}(D) + \sum_{ij} \begin{vmatrix} X_{ii}(D) & X_{ij}(D) \\ X_{ji}(D) & X_{jj}(D) \end{vmatrix} +$$

$$+ \sum_{ijk} \begin{vmatrix} X_{ii}(D) & X_{ij}(D) & X_{ik}(D) \\ X_{ji}(D) & X_{jj}(D) & X_{jk}(D) \\ X_{ki}(D) & X_{kj}(D) & X_{kk}(D) \end{vmatrix} + \ldots + \det X(D) \qquad (3.4)$$

Because, according to (2.5), we have

$$X(D) = \frac{F_1(D)c_1(D)F_n(D)c_n(D)\ldots F_2(D)c_2(D)}{\Lambda_1(D)\Lambda_n(D)\ldots\ldots\Lambda_2(D)}$$

then all $X_{jk}(D)$ have the divisor $\Lambda_0(D)$; and $K(D)$, after reduction to one denominator, takes the form: $K(D) = L(D)/\Lambda_0^{m_1}(D)$

where $L(D)$ is a polynomial. But the expression

$$\Lambda(D) = \det f(D) = \Lambda_0(D)\,[1 + K(D)] = \Lambda_0(D) + \frac{L(D)}{\Lambda_0^{m_1-1}(D)}$$

should be an integral rational function, because it is the characteristic determinant of the close circuit. Consequently

$$\frac{L(D)}{\Lambda_0^{m_1-1}(D)} = M(D)$$

where $M(D)$ is a polynomial; hence

$$L(D) = M(D)\Lambda_0^{m_1-1}(D), \quad K(D) = \frac{M(D)}{\Lambda_0(D)} \qquad (3.5)$$

This is the form $K(D)$ should have after calculation and possible reductions. The characteristic determinant of the closed circuit will be

$$\Lambda(D) = \Lambda_0(D)\,[1 + K(D)] = \Lambda_0(D) + M(D) \qquad (3.6)$$

Knowing it we may write the characteristic equation

$$\Lambda(D) = 0 \qquad (3.7)$$

The expression (3.6) makes it possible to investigate by ordinary rules the stability with the aid of the frequency criteria of A. V. Mikhaylov and Nyquist or the Routh-Hurwitz inequalities.

Since in the calculation of the characteristic determinant the external actions x and z are immaterial, we may obtain other expressions for K(D), interrupting the circuit after the second link, after the third etc.:

$$1 + K(D) = \det \left[E_{m_2} + X_2(D)X_1(D)X_n(D) \ldots X_3(D) \right] =$$
$$= \det \left[E_{m_3} + X_3(D)X_2(D)X_1(D)X_n(D) \ldots X_4(D) \right] = \qquad (3.8)$$

Us Usually it is most convenient to interrupt after the link α_j for which m_j is a minimum, preferably unity, since in this case the transmittance of the open circuit and the corresponding recurrent difference are obtained by scalars. In this case we have

$$K(D) = X_j(D) \ldots X_1(D)X_n(D) \ldots X_{j+1}(D)$$

4. Automatic Control of the Course of a Ship

As an example of the theory developed in the preceding sections, 1 to 3, let us consider the well known equations of the automatic control of a ship's course, which may be found, for example, in Basin (5) Vedrov (6), Weiss (7), Grammel (8), Mac-Call (9) and, Chalmers (10); these equations may also be applied to aircraft.

The equation of moments relative to the ship's vertical axis, passing through the center of gravity, and the equation of the lateral forces may, for small deviations from the rectilinear course, be written in the form

$$(T^2D^2 + UD)\theta + k\alpha - \beta = 0 \qquad (4.1)$$
$$-FD\theta + (SD + 1)\alpha - m\beta = 0$$

Here θ designates the yaw angle (Figure 3) - that is, the angle that the ship's longitudional axis makes with the given course; alpha is the angle of drift - that is, the angle that the longitudinal axis makes with the velocity vector v of the center of gravity; beta is the helm angle. The remaining letters (except D) designate coefficients which may be considered constant for a given velocity.

The equations of the control automat and of the rudder mechanism will be

$$= (L^2D^2 + MD + \ell)(x - \theta) - n$$
$$(V^2D^2 + WD)\beta = h\sigma \qquad (4.2)$$

Here sigma is the coordinate of the servomechanism; x is the command; L^2, M, , n designate parameters of the setting of the automat; V^2, W, h designate the parameters of the rudder mechanism.

According to the first equation the automat reacts in general to the yaw angle, angular velocity and acceleration, as registered by sensitive elements, and to the signal proportional to the helm angle as transmitted by feedback.

In order to decrease the number of constant parameters, let us introduce new variables t', ', ' by means of the ratios

$$t = \frac{T^2}{U}\, t', \quad \alpha = \frac{R}{S}\, \alpha', \quad \sigma = \frac{U^3 W}{T^4 h}\, \sigma', \quad \beta = \frac{U^2}{T^2}\, \beta' \qquad (4.3)$$

and note that

$$D = \frac{U}{T^2}\, D' \qquad \left(D' = \frac{d}{dt'}\right)$$

Transforming the equations we find

$$(D'^2 + D')\theta + k'\alpha' - \beta' = 0$$

$$-S'D'\theta + (S'D' + 1)\alpha' - m'\beta' = 0$$

$$\sigma' = (L'^2 D'^2 + M'D' + \ell')(x - \theta) - n'\beta'$$

$$(V'^2 D'^2 + D')\beta' = \sigma'$$

Here

$$k' = \frac{RT^2}{SU^2}\, k, \quad S' = \frac{U}{T^2}\, S, \quad m' = \frac{SU^2}{RT^2}\, m, \quad L'^2 = \frac{h}{UW}\, L^2$$

$$M' = \frac{T^2 h}{U^2 W}\, M, \quad \ell' = \frac{T^4 h}{U^3 W}, \quad n' = \frac{T^2 h}{UW}\, n, \quad V'^2 = \frac{U}{T^2 W}\, V^2$$

The nonprimed and primed constants enter the original and the transformed equations exactly in a similar way except that to the constants T, U, W, R, h of the first system correspond the values 1, 1, 1, S', 1 in the second. Therefore we may simply assume that the considered transformation is already performed and omitting the primes, set in the original equations

$$T = U = W = 1, \quad R = S, \quad h = 1.$$

Next solving formally equations (4.1) in , and equation (4.2) in , , we obtain matrix equations

$$\left\| \begin{matrix} \theta \\ \alpha \end{matrix} = X_1(D) \begin{matrix} \sigma \\ \beta \end{matrix} \right\|$$

$$\left\| \begin{matrix} \sigma \\ \beta \end{matrix} = -X_2(D) \begin{matrix} \theta \\ \alpha \end{matrix} \right\| + X_{2x}(D)_x$$

Here

$$X_1(D) = \frac{1}{\Delta_1(D)} \left\| \begin{matrix} 0 & SD + 1 - km \\ 0 & mD^2 + (m + S)D \end{matrix} \right\|$$

$$X_2(D) = \frac{L^2D^2 + MD +}{\Delta_2(D)} \left\| \begin{matrix} V^2D^2 + D & 0 \\ 1 & 0 \end{matrix} \right\|$$

$$X_{2x}(D) = \frac{L^2D^2 + MD +}{\Delta 2(D)} \left\| \begin{matrix} V^2D^2 + D \\ 1 \end{matrix} \right\|$$

$$\Delta_1(D) = SD^3 + (S + 1)D^2 + (Sk + 1)\, D; \quad \Delta_2(D) = V^2D^2 + D + n$$

The transmittance of the circuit open at the input of the second link, will be

$$X(D) = X_1(D)X_2(D) = \frac{L^2D^2 + MD + \ell}{\Delta_o(D)} \left\| \begin{matrix} SD + (1 - km) & 0 \\ mD^2 + (m + S)D & 0 \end{matrix} \right\|$$

where

$$\Delta_o(D) = \Delta_1(D)\Delta_2(D)$$

Hence

$$K(D) = (L^2D^2 + MD + \ell)\, [SD + (1-km)]/\Delta_o(D))$$

$$\Delta(D) = \Delta_o(D)[1 + K(D)] =$$

$$= k_oD^5 + k_1D^4 + k_2D^3 + k_3D^2 + k_4D + k_5$$

where

$$k_0 = SV^2$$

$$k_1 = S(V^2 + 1) + V^2$$

$$k_2 = S(L^2 + n + V^2k + 1) + V^2 + 1$$

$$k_3 = S(M + n + k) + L^2(1-km) + n + 1$$

$$k_4 = S(\ell + kn) + M(1-km) + n$$

$$K_5 = \ell(1-km)$$

It is easy to realize that the same expressions for K(D), Δ(D) are obtained if we open the circuit at the input of the first link.

For the investigation of stability let us confine ourselves to the particular case

$$L = M = V = 0$$

that is, we assume that the servomechanism reacts only to the yaw angle, and the inertia of the steering mechanism may be neglected.

Then the characteristic equation $\Delta(\quad) = 0$ may we written in the form

$$P(\lambda)\, n + Q(\lambda) \quad + R(\lambda) = 0$$

where

$$P(\lambda) = S\lambda^3 + (S + 1)\lambda^2 + (Sk + 1)$$

$$Q(\lambda) = S\lambda + (1 - km)$$

$$R(\lambda) = S\lambda^4 + (S + 1)\lambda^3 + (Sk + 1) \quad ^2$$

Assuming $\lambda = i\omega$ we separate in the characteristic equation the real and imaginary parts, and solving the two equations obtain for n and ℓ, we derive ratios of the form

$$n = \Lambda_n(\omega)/\Lambda(\omega)$$

$$\ell = \Lambda_\ell(\omega)/\Lambda(\omega)$$

These ratios may be considered as parametric equations of a certain curve N in the $n\ell$- plane. It divides the plane into regions in which the characteristic equation possesses one and the same number of roots with a

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positive real part [11]. One of these regions will be the region of stability (there the number of roots with a positive real part will be zero). In this case we have

$$\Lambda_1(\omega) = \omega^2\left[-S^2\omega^2 + S^2k + km(S+1) - 1\right]$$

$$\Lambda_2(\omega) = \omega^2\left[-S^2\omega^4 + (2S^2k - S^2 - 1)\omega^2 - (Sk+1)^2\right]$$

$$\Lambda(\omega) = -S(S+km)\omega^2 - (1-km)(Sk+1)$$

The shading of the curve was done according to Neymark's rules.

Singular straight lines corresponding to = 0,00 are the abscissa and the straight line at infinity.

If in the characteristic equation and in the equation

$$P'(\lambda)\,n + Q'(\lambda)\,\rho + R'(\lambda) = 0$$

we set $\lambda = \xi$ and solve for n, ρ, then we obtain a parametric equation of the discriminant curve (12):

where $n = \frac{\Gamma_1(\xi)}{\Gamma(\xi)}$ $\rho = \frac{\Gamma_2(\xi)}{\Gamma(\xi)}$

$\Gamma_1(\xi) =$ [illegible]

[illegible]

$\Gamma_2(\xi) =$ [illegible] $(S+1)\xi$ [illegible]

$\Gamma(\xi) =$ [illegible] $(1-$ [illegible] $)(S+1)\xi - (1-km)(Sk+1)$

Let us take, for instance, for the natural parameters of the ship the numerical values cited in chapter VIII of book by A. M. Basin (5).

Then we shall obtain the following values of the parameters of the equations (before the transformation of variables):

$T^2 = 0.6601$

$U = 2.007$

$R = 0.6009$

$S = 1.295$

$k = -0.5281$

$m = 0.2331$

After transformation we get:

$K = -0.04016$

$S = 3.938$

$m = 3.066$

The curve N and the discriminant curve for these parameters are represented in Figure 4.

The curve N and the positive part of the abscissa, shaded according to Neymark's rules, segregates the region of stability. The discriminat curve segregates the sub-region of aperiodical damping (shown by continuous shading).

The author expresses his gratitude to I. Z. Pirogov, who performed all the calculations necessary for the construction of the curves.

5. Gyroscopic Stabilizer

As a second example, for which the author is obliged to Ya. N. Roytenberg, let us consider the gyroscopic stabilizer, in his article (13), from which the author took the equations.

The instrument is intended to keep a platform, which may oscillate around a horizontal axis, in a horizontal position. It consists of a gyroscope with three degrees of freedom, the housing of which plays the role of a Cardan ring, while the external ring is rigidly joined to the platform, so that the axis of the platform is simultaneously the axis of the outer ring.

The shaft of the direct-current electric motor, which has to overcome the action of external forces on the platform, is coupled to this axis by means of gear transmission. The electric motor has an independent excitation, and the current in its armature is controlled by the amplifier. The voltage to the amplifier input is fed from a potentiometer, fixed to the gyroscope housing.

During the action of external disturbing forces on the platform, producing a moment relatively to the axis, the gyroscope precesses around the axis of its housing, and the potentiometer fixed to the housing feeds voltage to the input of the amplifier. The voltage after amplification is transmitted to the circuit of the armature of the motor, which develops a moment counteracting the moment of external forces.

The gyroscope besides controlling the motor, also performs the function of direct stabilization of the platform, because the currents in the amplifier and the armature of the electric motor, and therefore also the torque of the latter, has not yet reached the required value; the balancing of the external moment occurs not only on account of the moment of the motor, but also on account of the gyroscopic moment, developed as result of the precession of the gyroscope.

For a linear law of the potentiometer, the equations of the system, may be represented in the form

$$AD\omega - HD\beta - Ri = 0 \qquad \text{(gyroscopic}$$
$$H\omega + BD^2\beta = 0 \qquad \text{system)}$$

$$\mu\omega + (\tau D + 1)\, i - pI = 0 \quad \text{(motor}$$
$$-S\beta + (\sigma D + 1)I = 0 \qquad \text{(amplifier)}$$

Here ω is the angular velocity of the platform; β is the angle of precession of the gyroscope housing relative to the platform; i and I the currents in the circuit of the motor armature and in the amplifier; A is the moment of inertia of the platform and of the gyroscopic system relative to the platform axis together with the mentioned moment of inertia of the motor armature; B is the equatorial moment of inertia of the gyroscope; H is its kinetic moment; μ, τ, p, R, S, σ are the parameters of the motor and amplifier.

Taking as the first link of the circuit the gyroscopic system with the motor and taking as the second link the amplifier, we have the following matrix equations:

$$\begin{Vmatrix} \omega \\ \beta \\ i \end{Vmatrix} = X_1(D)I \qquad\qquad I = -X_2(D) \begin{Vmatrix} \omega \\ \beta \\ i \end{Vmatrix}$$

in which

$$X_1(D) = \frac{p}{\Delta_1 \cancel{(D)}} \begin{Vmatrix} BRD^2 \\ -HR \\ D(ABD^2 + H^2) \end{Vmatrix}$$

$$X_2(D) = \frac{S}{\Lambda_2(D)} \quad \| 0, -1, 0 \|$$

$$\Lambda_1(D) = AB\,[\tau D^4 + D^3 + (n + q^2\tau)D^2 + q^2 D],$$

$$\Lambda_2(D) = \sigma D + 1$$

$$n = F\mu/A, \quad q = H/\sqrt{AB}$$

The transmittance of the circuit open at the output of the second link will be:

$$X(D) = X_2(D)X_1(D)$$

$$= mH^2/\Lambda_o(D)$$

where

$$m = K\mu p/H, \quad \Lambda_o(D) = \Lambda_1(D)\Lambda_2(D).$$

Because $X(D)$ her is a scalar, then

$$K(D) = X(D)$$

and the characteristic determinant will be

$$\Lambda(D) = \Lambda_o(D)[1 + k(D)]$$

$$= \Lambda_o(D) + mH^2$$

or

$$\Lambda(D) = AB[\sigma\tau D^5 + (\sigma + \tau)D^4 + (1 + n\sigma + q^2\sigma\tau)D^3 + (n + q^2\sigma + q^2\tau)D^2 + q^2 D + mq^2]$$

6. Characteristic Equation of the Mofified System

As a generalization of the problem in section 3 let us consider the following problem: knowing the characteristic determinant $\Lambda_o(D)$ of a linear system:

$$f_o(D)y = 0$$

find the determinant $\Lambda(D)$ of the modified system

$$f(D)y = 0$$

Here, keeping in mind the application of frequency methods to the investigation of stability, one is required to determine the ratio $\Delta(i\omega)/\Delta_0(i\omega)$ by either computational or experimental means.

It is assumed that $f_0(D)$, $f(D)$ are square matrices of the same order n and that the matrix

$$\delta f(D) = f(D) - f_0(D)$$

has non-zero elements only in the rows with numbers $j_1 \ldots j_p$ and in columns with numbers $k_1 \ldots k_q$. Setting

$$u = \begin{Vmatrix} y_{k_1} \\ \cdot \\ \cdot \\ \cdot \\ y_{k_q} \end{Vmatrix}$$

and representing the second matrix equation in the form

$$f_0(D)y = - \quad f(D)y$$

we may substitute it by the equivalent system

$$f_0(D)y = \quad v, \quad u = \mathcal{E} y, \quad v = - X_2(D)u \qquad (6.1)$$

where is a matrix of the type n x p, in which the elements $\varepsilon_{j1,1}$, $\ldots \varepsilon_{jp,p}$ equal unity, and the remaining are zeros; further $\mathcal{E}$ is a matrix of the type q x n, in which the elements equal unity, and the remaining are zeros; and finally

$$X_2(D) = \begin{Vmatrix} \delta f_{j1,k1}(D) & \ldots & \delta f_{j1,\,kq}(D) \\ \ldots & \ldots & \ldots \\ \delta f_{jp,k1}(D) & \ldots & \delta f_{jp,\,kq}(D) \end{Vmatrix} \qquad (6.2)$$

The three equations (6.1) may be considered as the equations of the links of a closed circuit. The operational matrices of the links are $f_o(D)$, E_q, E_p and the characteristic determinant of the open circuit will be $\Delta_o(D)$. Solving the first equation for y, we:

$$\begin{aligned} y &= X_1'(D)\,v \\ u &= \mathcal{E}\,y \\ v &= -X_2(D)u \end{aligned} \qquad (6.3)$$

where

$$X_1'(D) = \frac{1}{\Delta_o(D)} \left\| \begin{matrix} F^{(0)}_{1,\,j1}(D) & \dots & F^{(0)}_{1,\,jp}(D) \\ \dots & \dots & \dots \\ F^{(0)}_{n,\,j1}(D) & \dots & F^{(0)}_{n,\,jp}(D) \end{matrix} \right\|$$

and $F_{jk}(D)$ are elements of the matrix $F_o(D)$ which is added to $f_o(D)$.

The determinant of the closed circuit is, according to (3.6),

$$\Delta(D) = \Delta_o(D)[1 + K(D)]$$

while $1 + K(D)$ equals the determinant of any one of the two matrices

$$E_q + X_1'(D)X_2(D), \quad E_p + X_2(D)\mathcal{E}X_1'(D)$$

which represent the recurrent differences obtained upon disconnecting the cirucit after link α_μ or after link α_v.

Setting $X_1(D) = \mathcal{E}\,X_1'(D)$ or

$$X_1(D) = \frac{1}{\Delta_0(D)} \left\| \begin{matrix} F^{(0)}_{k1,\,j1}(D) & \dots & F^{(0)}_{k1,\,jp}(D) \\ \dots & \dots & \dots \\ F^{(0)}_{kq,\,j1}(D) & \dots & F^{(0)}_{kq,\,jp}(D) \end{matrix} \right\| \qquad (6.4)$$

we shall represent the recurrent differences in a more simple form

$$E_q + X_1(D)X_2(D), \quad E_p + X_2(D)X_1(D) \tag{6.5}$$

and then

$$\Delta(D) = \Delta_0(D)\det[E_q + X_1(D)X_2(D)] = \Delta_0(D)\det[E_p + X_2(D)X_1(D)] \tag{6.6}$$

We may also eliminate y from the system (6.3). The equations thus obtained are

$$u = X_1(D)v, \qquad v = -X_2(D)u \tag{6.7}$$

and should be considered as the equations of a two-link circuit.

Interrupting the circuit at the output of link α_u, one can measure experimentally the elements of the matrix $E_q + X_1(i\omega)X_2(i\omega)$ by comparing the sinusoidal signals supplied to the input of link α_v with the response signals received at the output of link α_u. The closed circuit and the position of cut-off are represented in a schematic diagram (Figure5), where the link $\quad_u$, making up part of the unmodified system α_o, is represented inside the latter. The link α_v represents the aggregate of those elements by which the system α_o differs from the modified system .

7. Multi-Loop Closed Circuits

The theorems obtained in sections 3 and 6 above may be applied to the calculation of the characteristic determinants and to the frequency investigations of stability of complex multi-loop circuits. For this purpose it is necessary first, by elimination of some elements and internal connections, to transform the given system into a passive system the stability of which for any complexity is not required in an analytic check. Next it is necessary to return to the original system, by applying in any sequence and combinations the blocking method; that is, the joining of one internal loop or another to one link with a matrix coordinate (see section 3), and the method of successive reconstitution, stage by stage of rejected elements and connections (see section 6). At every stage it is necessary as usual to check the zeros of the determinants corresponding to the recurrent differences [9] (Chapters VIII) which may be found both by computation from the equations and by experiments - namely, by forming the frequency characteristics of real elements or groups of elements.

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FIGURES

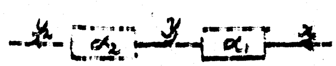

Figure 1.

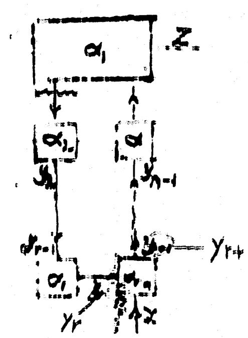

Figure 2.

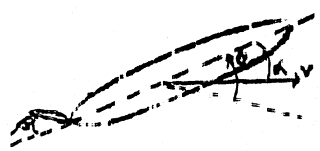

Figure 3.

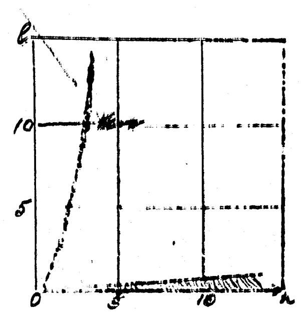

Figure 4.

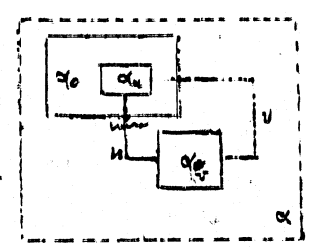

Figure 5

- E N D -